Bounds for smoothness of refinable functions

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***Abstract*— The VILLEMOES machine can be used to compute the SOBOLEV smoothness of a refinable function. We start with presenting this technique. It involves the computation of the spectral radius of a special matrix which has at least quadratic time complexity with respect to the refinement mask size. For the one-dimensional case we deduce by linear algebra some simple estimates which require only a few basic operations on the mask coefficients with a total of linear time complexity. For orthogonal DAUBECHIES and biorthogonal CDF wavelet generators the estimates are compared to the known regularities.**

I. INTRODUCTION

Smoothness[1] of functions is often measured in terms of HOELDER continuity and SOBOLEV smoothness. It is a difficult topic how to compute such smoothness measurements from a known refinement mask of a refinable function but several authors created practical techniques for this purpose [Vil94], [Eir92], [Con90], [Dau92].

The VILLEMOES machine [Vil94], [BDM00] is a popular method for computing the global SOBOLEV smoothness of a refinable function. It consists mainly of the computation of the largest eigenvalue of a so called transition matrix. It is easy structured for one-dimensional problems and fast enough to determine the smoothness of single given refinable functions. However, for automatic generation of smooth refinable functions e.g. by iterative correction it is too time-consuming to start the VILLEMOES machine for each iteration.

By linear algebra we will derive some simple estimates from the VILLEMOES theory which involve only a few basic operations. Some estimates show theoretical limits of the smoothness depending on the length of the filter mask and one allows for verification whether a constructed wavelet is smooth enough.

II. DEFINITIONS

The VILLEMOES machine is a technique which computes the SOBOLEV smoothness of a refinable function straight from the coefficients of the refinement mask.

Definition 1: The vector h with $h \in \mathbb{R}^{\mathbb{Z}}$ and a finite number of non-zero entries and $\sum_{k\in\mathbb{Z}} h_k = 1$ is called a *refinement mask* for the function φ if

$$\varphi(t) = 2 \cdot \sum_{j\in\mathbb{Z}} h_j \varphi(2t - j) \tag{1}$$

holds. Vice versa the function φ is called *refinable* with respect to the mask h. For $\nu = \min\{j \in \mathbb{Z} : h_j \neq 0\}$ and $\kappa = \max\{j \in \mathbb{Z} : h_j \neq 0\}$ define the index set $\mathcal{I} = \{\nu, \dots, \kappa\}$ which is the *support* of the mask h.

To be able to state VILLEMOES result about the smoothness of refinable functions we need the notion of a RIESZ basis, especially a RIESZ basis of integer translates of a refinable function.

Definition 2: A sequence $(f_k)_{k\in\mathbb{Z}}$ of linear independent functions f_k from a HILBERT space $\mathbb{H}$ is called a RIESZ *basis* of $\mathbb{H}$ if the set of linear combinations of f_k is dense in $\mathbb{H}$ and the norm in $\mathbb{H}$ is equivalent to the ℓ_2 norm of expansion coefficient sequences, that is

$$\exists (C_1, C_2) \in \mathbb{R}_+^2 \quad \forall a \in \mathbb{R}^{\mathbb{Z}}$$
$$C_1 \cdot \|a\|_2^2 \leq \left\| \sum_{k\in\mathbb{Z}} a_k f_k \right\|_{\mathbb{H}}^2 \leq C_2 \cdot \|a\|_2^2$$

.

Definition 3: Let φ_k be φ translated by k, that is

$$\forall t \in \mathbb{R} : \varphi_k(t) = \varphi(t - k).$$

Definition 4: If the sequence of translates $(\varphi_k)_{k\in\mathbb{Z}}$ from a HILBERT space $\mathbb{H}$ forms a RIESZ basis of the closure of its linear span, we say that φ has the RIESZ basis property $\mathcal{B}(\varphi)$, that is

$$\mathcal{B}(\varphi) \Leftrightarrow \exists (C_1, C_2) \in \mathbb{R}_+^2 \quad \forall a \in \mathbb{R}^{\mathbb{Z}}$$
$$C_1 \cdot \|a\|_2^2 \leq \left\| \sum_{k\in\mathbb{Z}} a_k \varphi_k \right\|_{\mathbb{H}}^2 \leq C_2 \cdot \|a\|_2^2 .$$

For some considerations it is easier to switch to the FOURIER space. A FOURIER transform maps a vector h to a trigonometric polynomial $\widehat{h}$.

[1] the term *regularity* is avoided according to [SN97]

Definition 5: Given a mask h define the trigonometric polynomial $\widehat{h}$:

$$\widehat{h}(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{-ik\xi}.$$

Definition 6: Given a mask h define the adjoint filter h^*:

$$\forall j \in \mathbb{Z} : h^*_j = \overline{h_{-j}}.$$

It holds $\forall\xi \in \mathbb{R} : \overline{\widehat{h}(\xi)} = \widehat{h^*}(\xi)$. For the convolution of a mask h with its adjoint holds

$$\widehat{h * h^*} = \widehat{h} \cdot \overline{\widehat{h}} = \left|\widehat{h}\right|^2.$$

Definition 7: For a given mask h with finite support $\mathcal{I}$ the matrix P_h with $P_h \in \mathbb{R}^{\mathcal{I}^2}$ is defined as

$$\begin{aligned} P_h &= (h_{2j-k})_{(j,k)\in\mathcal{I}^2} \\ &= \begin{pmatrix} h_\nu & & & & & \\ h_{\nu+2} & h_{\nu+1} & h_\nu & & & \\ h_{\nu+4} & h_{\nu+3} & h_{\nu+2} & h_{\nu+1} & h_\nu & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & h_\kappa & h_{\kappa-1} & h_{\kappa-2} & h_{\kappa-3} & h_{\kappa-4} \\ & & & h_\kappa & h_{\kappa-1} & h_{\kappa-2} \\ & & & & & h_\kappa \end{pmatrix} \end{aligned}$$

and the special matrix $2P_{h*h^*}$ is called the *transition matrix* of h [SN97].

The matrix P_h describes a convolution with subsequent subsampling by two. If one evaluates the refinement equation (1) for different integers t one discovers a dependency that allows for computation of the values of the refinable function φ at integer values:

$$\begin{pmatrix} \varphi(\nu) \\ \varphi(\nu+1) \\ \vdots \\ \varphi(\kappa) \end{pmatrix} = 2P_h \cdot \begin{pmatrix} \varphi(\nu) \\ \varphi(\nu+1) \\ \vdots \\ \varphi(\kappa) \end{pmatrix}.$$

The structure of the eigenvalue spectrum of such matrices is the key for measuring the smoothness of wavelets.

III. VILLEMOES MACHINE

Most commonly the smoothness of a function is described by its membership in a space of functions of certain degree of smoothness, where the family of SOBOLEV spaces and the family of HOELDER spaces are the most popular ones.

To describe spaces with fractional degrees of smoothness we make use of the FOURIER transform. The FOURIER transform of a function f is denoted by $\widehat{f}$. Note that we use the same notation for the fourier symbol of a discrete vector as well as for the fourier transform of a real function since there is a certain analogy.

Further on we need $S(\mathbb{R})$ which is the SCHWARZ space consisting of fast decaying arbitrarily often differentiable functions and its dual space $S'(\mathbb{R})$ which is the collection of all complex-valued tempered distributions on $\mathbb{R}$.

For more compact notation we will use the FOURIER multiplicator ϑ^s_q.

$$\vartheta^s_q(\xi) = (1 + |\xi|^q)^{s/q}$$

The key component of computing smoothness measures for refinable functions is the following operation defined for a mask m:

1) Extract the factor $\cos^2\frac{\xi}{2}$ as often as possible from $\widehat{m}(\xi)$. That is choose K such that
$$\widehat{m}(\xi) = \left(\cos\frac{\xi}{2}\right)^{2K} \cdot \widehat{h}(\xi)$$
where h is a mask without a double zero at π, i.e. $\widehat{h}(\pi) \neq 0$ or $\widehat{h}'(\pi) \neq 0$. Note that $\cos^2\frac{\xi}{2}$ corresponds to the coefficient vector $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.
2) Set-up the matrix P_h and compute the absolute value of its largest eigenvalue. This is denoted by the *spectral radius* $\varrho(P_h)$.
3) The result of the operation is
$$\begin{aligned} M_m &= 2K - \log_2 \varrho(2P_h) \\ &= 2K - 1 - \log_2 \varrho(P_h). \end{aligned}$$

A. HOELDER *continuity*

The HOELDER-ZYGMUND function spaces $\mathcal{C}^s(\mathbb{R})$ with $s \in \mathbb{R}$ and $s \geq 0$ contains all functions that are up to $\lceil s \rceil$ times differentiable and some more functions. A characterization can be given using a smooth dyadic resolution of the unity $\{\psi_j\}_{j\in\mathbb{N}_0}$ and the operator D which "redirects" the multiplication with ψ_j to the FOURIER representation ([Tri92], pages 14-17):

$$\mathcal{C}^s(\mathbb{R}) = \left\{ \begin{array}{l} f \in S'(\mathbb{R}) : \\ \quad \sup_{t\in\mathbb{R},j\in\mathbb{N}_0} 2^{js} |\psi_j(D)f(t)| < \infty \end{array} \right\}$$

Lemma 1:

$$\left\{ f \in S'(\mathbb{R}) : \vartheta^s_1 \cdot \widehat{f} \in L_1(\mathbb{R}) \right\} \subset \mathcal{C}^s(\mathbb{R}).$$

Proof: For all $j \in \mathbb{N}_0$ and $t \in \mathbb{R}$ it holds for a smooth dyadic resolution of unity $\{\psi_j\}_{j\in\mathbb{N}_0}$ like those described in [Tri92], page 15 that

$$\begin{aligned} 2^{js}\left|\psi_j(D)f(t)\right| &\leq 2^{js}\int\left|\psi_j(\xi)\widehat{f}(\xi)\right|\mathrm{d}\,\xi \\ &\leq 2^s\int(1+|\xi|)^s\left|\widehat{f}(\xi)\right|\mathrm{d}\,\xi. \end{aligned}$$

and thus it is true also for the supremum. ■

An estimate of the HOELDER continuity for refinable functions in terms of their refinement mask was derived from this embedding by Conze and Raugi [CR90], [Con90], for a summary see [Dau92]. The estimate can be made more simple in the case that $\widehat{m}$ is a positive function, that is $\forall\xi \in \mathbb{R} : \widehat{m}(\xi) \geq 0$.

Theorem 1: Given the mask m decide:

1) If $\widehat{m}$ is positive, set $s_0 = M_m$.
2) If $\widehat{m}$ is not positive, set $s_0 = \frac{1}{2}\left(M_{m*m^*} - 1\right)$.

Let φ be the refinable function associated with the mask m. Then it holds

$$\forall s \in \mathbb{R} : s < s_0 \Rightarrow \varphi \in \mathcal{C}^s(\mathbb{R})$$

B. SOBOLEV *smoothness*

A SOBOLEV space $W_p^s(\mathbb{R})$ for $s \in \mathbb{N}_0$ is defined as the space of distributions from $S'(\mathbb{R})$ whose derivatives up to order s are in $L_p(\mathbb{R})$. This idea was generalized to the SOBOLEV spaces $H_p^s(\mathbb{R})$ of fractional order s [Tri92]. We restrict ourselves to $H_2^s(\mathbb{R})$ which allows for a characterization that was used by VILLEMOES to explore the smoothness of refinable functions.

Definition 8: The SOBOLEV function space $H_2^s(\mathbb{R})$ is defined as

$$H_2^s(\mathbb{R}) = \left\{ f \in S'(\mathbb{R}) : \vartheta_2^s \cdot \widehat{f} \in L_2(\mathbb{R}) \right\}.$$

The SOBOLEV smoothness of a refinable function can be characterized similarly to Theorem 1 ([Vil93], theorem 2.3).

Theorem 2: Given the mask m let $s_0 = \frac{1}{2}M_{m*m^*}$. Then it holds

1) $\forall s \in \mathbb{R} : s < s_0 \Rightarrow \varphi \in H_2^s(\mathbb{R})$
2) $\forall s \in \mathbb{R} : \mathcal{B}(\varphi) \wedge \varphi \in H_2^s(\mathbb{R}) \Rightarrow s < s_0$

that means s_0 can be regarded as an accurate measurement of the smoothness of φ.

IV. SIMPLE ESTIMATES

Theorem 1 and Theorem 2 states that the smoothness of a refinable function depends on the number of factors $\left(1+e^{-i\xi}\right)$ in $\widehat{m}(\xi)$ and on the remaining factor $\widehat{h}(\xi)$. More precisely the spectral radius of either P_h or P_{h*h^*} is the critical quantity. The number of factors $\left(1+e^{-i\xi}\right)$ is easy to handle normally, but the largest eigenvalue of P_h is not. Thus we will focus on the remaining mask h and $\varrho(P_h)$.

Remark 1: As asserted in Definition 1 the sum of the coefficients of the filter mask h is always 1 $\left(\widehat{h}(0) = 1\right)$. Hence the sum of the coefficients of $h * h^*$ also equals 1 $\left(\widehat{h * h^*}(0) = \left|\widehat{h}(0)\right|^2 = 1\right)$. According to Theorem 1 and Theorem 2 we will consider only matrices P of positive filter polynomials and their filter coefficients will always sum up to 1.

Lemma 2: The first and the last non-zero mask coefficient, h_ν and h_κ respectively, are eigenvalues of the matrix P_h.

Proof: Expand the determinant $\det(P_h - \lambda I)$ for the top and the bottom row. ■

There are some simple general ways of estimating the spectral radius of a matrix. E.g. $\varrho(P_h) \leq \|P_h\|$ holds for any compatible matrix norm. We will show that such estimates are too weak in some cases. This should motivate the search for stronger estimates as presented at the end of this section.

The following statements show that the column and row sum matrix norms are bounded from below. Thus estimates based on these norms can not benefit from the fact that longer filters allow smaller spectral radii.

Lemma 3:
1) If $\kappa - \nu$ is even, then the row sum norm of the matrix P_h is at least 1.
2) If $\kappa - \nu$ is odd, then the row sum norm of the matrix P_h is at least $\frac{1}{2}$.

Proof: Case $2 \mid (\kappa - \nu)$:
The $\frac{\nu+\kappa}{2}$th row of P_h which is the center row consists of all mask coefficients $h_\nu, \ldots, h_\kappa$ thus

$$\begin{aligned} \|P_h\|_\infty &= \max_{j\in\mathcal{I}}\sum_{k\in\mathcal{I}}\left|(P_h)_{j,k}\right| \\ &\geq \sum_{k\in\mathcal{I}}\left|(P_h)_{\frac{\nu+\kappa}{2},k}\right| &&= \sum_{k\in\mathcal{I}}|h_k| \\ &\geq \left|\sum_{k\in\mathcal{I}}h_k\right| &&= 1 \end{aligned}$$

Case $2 \nmid (\kappa - \nu)$:
The $\frac{\nu+\kappa-1}{2}$th row of P_h consists of all mask coefficients except h_κ and the $\frac{\nu+\kappa+1}{2}$th row of P_h consists of all mask coefficients except h_ν and thus

$$\begin{aligned}
\|P_h\|_\infty &\geq \max_{j\in\left\{\frac{\nu+\kappa-1}{2},\frac{\nu+\kappa+1}{2}\right\}} \sum_{k\in\mathcal{I}} \left|(P_h)_{j,k}\right| \\
&= \max\{|h_\nu|, |h_\kappa|\} + \sum_{k\in\mathcal{I}\setminus\{\nu,\kappa\}} |h_k| \\
&\geq \frac{1}{2}(|h_\nu| + |h_\kappa|) + \sum_{k\in\mathcal{I}\setminus\{\nu,\kappa\}} |h_k| \\
&\geq \frac{1}{2}\left(\sum_{k\in\mathcal{I}} |h_k| + \sum_{k\in\mathcal{I}\setminus\{\nu,\kappa\}} |h_k|\right) \\
&\geq \frac{1}{2}\left(1 + \sum_{k\in\mathcal{I}\setminus\{\nu,\kappa\}} |h_k|\right) \\
&\geq \frac{1}{2}
\end{aligned}$$

■

The column sum norm might be better suited.

Lemma 4: The column sum norm of the matrix P_h is at least $\frac{1}{2}$.

Proof: For $\nu = \kappa$ it must be $h_\nu = 1$ (Definition 1) and thus $\|P_h\|_1 = 1$. For $\nu < \kappa$ the matrix P_h has at least two columns. We consider the first two:

$$\begin{aligned}
\|P_h\|_1 &= \max_{k\in\mathcal{I}} \sum_{j\in\mathcal{I}} \left|(P_h)_{j,k}\right| \\
&= \max_{k\in\{\nu,\nu+1\}} \sum_{j\in\mathcal{I}} \left|(P_h)_{j,k}\right| = \max_{k\in\{0,1\}} \sum_{j\in(k+2\mathbb{Z})} |h_j| \\
&\geq \frac{1}{2} \sum_{k\in\{0,1\}} \sum_{j\in(k+2\mathbb{Z})} |h_j| \\
&\geq \frac{1}{2} \sum_{j\in\mathcal{I}} |h_j| \\
&\geq \frac{1}{2} \left|\sum_{j\in\mathcal{I}} h_j\right| = \frac{1}{2}
\end{aligned}$$

■

It is clear that long filters allow for at least the smoothness of short filters simply because long filters have additional degrees of freedom compared with short filters. The next statement quantifies this obserservation and gives a theoretical limit of the smoothness for a refinable function depending on the length ($\#\mathcal{I} = \kappa - \nu + 1$) of the mask.

Lemma 5: The spectral radius of the matrix P_h is always at least $\frac{1}{\#\mathcal{I}}$.

$$\varrho(P_h) \geq \frac{1}{\#\mathcal{I}}$$

Proof: We make use of the fact that the diagonal of P_h consist of all coefficients of the mask. We use the index set $\mathcal{I}$ for the eigenvalues λ_j, too, although the eigenvalues do not correspond one-to-one to the mask coefficients.

$$\begin{aligned}
\#\mathcal{I} \cdot \max_{j\in\mathcal{I}} |\lambda_j| &\geq \sum_{j\in\mathcal{I}} |\lambda_j| \\
&\geq \left|\sum_{j\in\mathcal{I}} \lambda_j\right| = |\operatorname{trace}(P_h)| \\
&= \left|\sum_{j\in\mathcal{I}} h_j\right| = 1
\end{aligned}$$

■

However the estimate of the smoothness depending on the mask can be refined using $\operatorname{trace} P_h^2$ instead of $\operatorname{trace} P_h$. More generally we observe that if P_h has eigenvalues $\lambda_\nu, \lambda_{\nu+1}, \ldots, \lambda_\kappa$ then P_h^n has eigenvalues $\lambda_\nu^n, \lambda_{\nu+1}^n, \ldots, \lambda_\kappa^n$. Thus $\operatorname{trace}(P_h^n) = \sum_{j\in\mathcal{I}} \lambda_j^n$. It is $P_h \cdot x = (h * x) \downarrow 2$ where $y \downarrow 2$ denotes the subsampling of y by a factor of 2 (See the appendix for further details.).

We are interested in a similar characterization for P_h^n.

Lemma 6:

$$P_h^n \cdot x = \left(h \uparrow 2^{n-1} * \ldots * h \uparrow 2 * h * x\right) \downarrow 2^n$$

Proof: We use induction over n. First we verify that

$$P_h^0 \cdot x = x = x \downarrow 2^0$$

For the induction step we need (4) of Lemma 10 of the appendix:

$$\begin{aligned}
P_h^n \cdot x &= \left(h \uparrow 2^{n-1} * \ldots * h * x\right) \downarrow 2^n \\
P_h^{n+1} \cdot x &= P_h^n \cdot P_h \cdot x \\
&= \left(h \uparrow 2^{n-1} * \ldots * h * (h * x) \downarrow 2\right) \downarrow 2^n \\
&\overset{(4)}{=} (h \uparrow 2^n * \ldots * h \uparrow 2 * h * x) \downarrow 2^{n+1}.
\end{aligned}$$

■

For simplification we will call the result of the convolution cascade H^n. It has support $\{(2^n - 1)\nu, \ldots, (2^n - 1)\kappa\}$.

$$H^n = h \uparrow 2^{n-1} * \ldots * h \uparrow 2 * h$$

With this notion we can characterize P_h^n using convolution and downsampling

$$P_h^n \cdot x = (H^n * x) \downarrow 2^n$$

and from this we can derive the matrix representation

$$P_h^n = \left(H^n_{2^n j - k}\right)_{(j,k)\in\mathcal{I}^2}.$$

We realize that the trace of P_h^n is essentially a sum of selected coefficients of H^n so in the next step we will explicitly compute the coefficients of H^n. Note that due to Definition 1 the mask H^n is an infinite vector with finite support.

Lemma 7: With the index set

$$\mathcal{J}_j^n = \left\{a \in \mathbb{Z}^n : a_0 + 2a_1 + \cdots + 2^{n-1}a_{n-1} = j\right\}$$

it holds that

$$(H^n)_j = \sum_{a\in\mathcal{J}_j^n} \prod_{l=0}^{n-1} h_{a_l} \tag{2}$$

.

Proof: The convolution of some finitely supported signals $x_0, \ldots, x_{n-1}$ that is $y = x_0 * \cdots * x_{n-1}$ can be computed component-wise as

$$y_j = \sum_{\substack{b\in\mathbb{Z}^n \\ b_0+\cdots+b_{n-1}=j}} \prod_{l=0}^{n-1} (x_l)_{b_l}.$$

For $x_l = h \uparrow 2^l$, i.e.

$$(x_l)_k = \begin{cases} h_{k/2^l} & : k \equiv 0 \mod 2^l \\ 0 & : \text{else} \end{cases}$$

and $b_l = 2^l a_l$ we obtain the claim. ■

Using the explicit representation of H^n the trace of P_h^n can be computed by

$$\begin{aligned}\operatorname{trace}(P_h^n) &= \sum_{j\in\mathcal{I}} H^n_{(2^n-1)j} \\ &= \sum_{j\in(2^n-1)\mathcal{I}} H^n_j \\ &= \sum_{j\in(2^n-1)\mathcal{I}} \sum_{a\in\mathcal{J}_j^n} \prod_{l=0}^{n-1} h_{a_l}\end{aligned}$$

and because of the finite support of h (Definition 1)

$$= \sum_{j\in(2^n-1)\mathbb{Z}} \sum_{a\in\mathcal{J}_j^n} \prod_{l=0}^{n-1} h_{a_l}.$$

For fixed n the index sets $\mathcal{J}_j^n$ are disjoint with respect to j. Thus the sums can be merged using a new index set $\mathcal{K}_0^n$. We want to introduce a more generic definition for $\mathcal{K}_k^n$:

$$\mathcal{K}_k^n = \bigcup_{j\in(2^n-1)\mathbb{Z}} \mathcal{J}_{j+k}^n$$

$$\operatorname{trace}(P_h^n) = \sum_{a\in\mathcal{K}_0^n} \prod_{l=0}^{n-1} h_{a_l}$$

This representation can still be improved for more efficient computation. We note that the set $\mathcal{K}_k^n$ is (2^n-1)-periodic, i.e. $\mathcal{K}_k^n + (2^n-1)\mathbb{Z}^n = \mathcal{K}_k^n$. The following identities may illustrate that:

$$\begin{aligned}\mathcal{K}_k^n &= \bigcup_{j\in\mathbb{Z}} \mathcal{J}^n_{(2^n-1)j+k} \\ &= \{a \in \mathbb{Z}^n : \\ &\qquad a_0 + \cdots + 2^{n-1}a_{n-1} \equiv k \mod (2^n-1)\} \\ &= \mathcal{J}_k^n + (2^n-1)\mathbb{Z}^n.\end{aligned}$$

We can use the periodicity to reduce the mask h to length (2^n-1). To analyse this we will partition $\mathcal{K}_k^n$ into the coarse grid $(2^n-1)\mathbb{Z}^n$ and the set $\mathcal{M}_k^n$ of the multi-indices within one grid cell.

$$\mathcal{M}_k^n = \mathcal{K}_k^n \cap \{0, \ldots, 2^n-2\}^n$$

Therefore the partition of $\mathcal{K}_k^n$ is

$$\mathcal{K}_k^n = \mathcal{M}_k^n + (2^n-1)\mathbb{Z}^n.$$

Now the trace of P_h^n can be computed more efficiently.

Lemma 8: With the operator S_k^n that sums up equidistant components of a vector, more precisely

$$S_k^n h = \sum_{j\in\mathbb{Z}} h_{k+(2^n-1)j}$$

it holds that

$$\operatorname{trace}(P_h^n) = \sum_{a\in\mathcal{M}_0^n} \prod_{l=0}^{n-1} S_{a_l}^n h.$$

Proof:

$$\begin{aligned}\operatorname{trace}(P_h^n) &= \sum_{a\in\mathcal{K}_0^n}\prod_{l=0}^{n-1} h_{a_l}\\ &= \sum_{a\in\mathcal{M}_0^n}\sum_{b\in(2^n-1)\mathbb{Z}^n}\prod_{l=0}^{n-1} h_{a_l+b_l}\\ &= \sum_{a\in\mathcal{M}_0^n}\prod_{l=0}^{n-1}\sum_{j\in(2^n-1)\mathbb{Z}} h_{a_l+j}\\ &= \sum_{a\in\mathcal{M}_0^n}\prod_{l=0}^{n-1} S_{a_l}^n h\end{aligned}$$

■

From the definition of $\mathcal{M}_k^n$ follows that for each choice of $a_1,\dots,a_{n-1}$ there is exactly one matching a_0, thus $\#\mathcal{M}_k^n = (2^n-1)^{n-1}$ which grows rather fast for increasing n. A discrete FOURIER transformation can speed up the computation, but the time consumed will still grow exponentially with respect to n.

That is why this formula is only useful for small n. Especially for $\operatorname{trace}(P_h^2)$ it turns out to be very handy. We will concentrate on this case for the rest of this paper. It is

$$\begin{aligned}\mathcal{M}_0^2 &= \left\{(a,b)\in\{0,1,2\}^2 : 0\equiv a+2b \mod 3\right\}\\ &= \left\{(a,b)\in\{0,1,2\}^2 : 0\equiv a-b \mod 3\right\}\\ &= \{(0,0),(1,1),(2,2)\}\end{aligned}$$

and thus

$$\operatorname{trace}(P_h^2) = (S_0^2h)^2 + (S_1^2h)^2 + (S_2^2h)^2.$$

Theorem 3: For a given mask h with finite support $\mathcal{I}$ let $y_j = S_j^2 h$ and $B_h = \sqrt{y_0^2+y_1^2+y_2^2}$. Then a lower bound for the spectral radius is given by

$$\frac{1}{\sqrt{\#\mathcal{I}}}\cdot B_h \le \varrho\left(P_h\right).$$

If the eigenvalues of P_h are all real then there is a simple upper bound:

$$\varrho\left(P_h\right) \le B_h.$$

Proof:

1)

$$\begin{aligned}\#\mathcal{I}\cdot\max_{j\in\mathcal{I}}|\lambda_j|^2 &\ge \sum_{j\in\mathcal{I}}|\lambda_j|^2\\ &\ge \left|\sum_{j\in\mathcal{I}}\lambda_j^2\right|\\ &= \left|\operatorname{trace}(P_h^2)\right| = B_h^2\end{aligned}$$

2)

$$\begin{aligned}\max_{j\in\mathcal{I}}|\lambda_j|^2 &\le \sum_{j\in\mathcal{I}}|\lambda_j|^2\\ &= \sum_{j\in\mathcal{I}}\lambda_j^{\,2} \quad = B_h^2\end{aligned}$$

■

Remark 2: One might hope that the eigenvalues of matrices of the form P_{h*h^*} are always real. The example $h = (2,0,0,-1)$ disproves this assumption. It is $h*h^* = (-2,0,0,5,0,0,-2)$ and P_{h*h^*} has the eigenvalues $\pm1\pm3i,-2,-2,5$.

Indeed there is a family of filters h which lead to a constant value of B_{h*h^*} according to Theorem 3 while the spectral radius of P_{h*h^*} is not bounded. Such a family is $\{(1+x,0,0,-x) : x\in\mathbb{R}\}$.

Remark 3: One might also assume that the existence of a *complementary filter* g (i.e. a filter g such that h and g allow for perfect reconstruction, see [DS98] for details), already implies that all eigenvalues of P_{h*h^*} are real. This is also not true since for $h=(2,0,0,-1), g=(0,0,1,0)$ the filter g is complementary to h.

Whether the spectral radius is closer to the upper bound or closer to the lower bound depends on the distribution of the eigenvalues of the matrix P_h. In the case that the eigenvalues have similar magnitude the spectral radius will be close to the lower bound. If there are only a few large eigenvalues and many small ones then the spectral radius will be close to the upper bound.

A simple lower estimate for the spectral radius that does not depend on the filter coefficients is given by

Lemma 9:

$$\varrho\left(P_h\right) \ge \frac{1}{\sqrt{3\cdot\#\mathcal{I}}}.$$

Proof: We derive this from Theorem 3 using the inequality of quadratic and arithmetic mean

$$\begin{aligned}\sqrt{\frac{1}{3}(y_0^2+y_1^2+y_2^2)} &\ge \frac{1}{3}(y_0+y_1+y_2)\\ \frac{1}{\sqrt{3}}\cdot B_h &\ge \frac{1}{3}\end{aligned}$$

and the last holds because

$$y_0+y_1+y_2 = \sum_{j\in\mathcal{I}} h_j = 1$$

due to Remark 1. ■

We will now consider an optimization for estimating the SOBOLEV smoothness of φ. According to Theorem 2 we have to process $h * h^*$ instead of the pure filter mask h to that end. Then $B_{h*h^*} = \sqrt{\sum_{j=0}^{2} \left(S_j^2(h*h^*)\right)^2}$. This can be further simplified thus avoiding the need for an explicit convolution $h*h^*$. With y_j as defined in Theorem 3 and

$$\begin{aligned} p_1 &= y_0 + y_1 + y_2 = 1 \\ p_2 &= y_0^2 + y_1^2 + y_2^2 \end{aligned}$$

we obtain

$$\begin{aligned} S_0^2(h*h^*) &= y_0 y_0 + y_1 y_1 + y_2 y_2 = p_2 \\ S_1^2(h*h^*) &= y_0 y_1 + y_1 y_2 + y_2 y_0 = \frac{p_1^2 - p_2}{2} \\ S_2^2(h*h^*) &= y_0 y_2 + y_1 y_0 + y_2 y_1 = \frac{p_1^2 - p_2}{2} \end{aligned}$$

and thus

$$\begin{aligned} B_{h*h^*} &= \sqrt{p_2^2 + 2 \cdot \left(\frac{1-p_2}{2}\right)^2} \\ &= \sqrt{\frac{3}{2}\left(p_2 - \frac{1}{3}\right)^2 + \frac{1}{3}} \end{aligned}$$

V. EXAMPLES

We will now compare our simple estimates with the exact regularities provided by Theorem 2 for two standard families of wavelet bases. The considered wavelet bases have filter polynomials that are not positive in general thus the HOELDER smoothness estimate according to Theorem 1 is derived from the SOBOLEV smoothness. Hence we only consider estimates of the SOBOLEV smoothness. The orthogonal DAUBECHIES wavelets as well as the biorthogonal COHEN-DAUBECHIES-FEAUVEAU wavelets (CDF) are chosen because they can be automatically constructed also for high orders (see [Dau92], chapters 6.1 and 8.3.4). The considered filter masks lead to transition matrices with real eigenvalues and thus both estimates of Theorem 3 can be applied.

The complete algorithm for estimating the SOBOLEV smoothness is

1) Let m be the filter mask.
2) Divide $\widehat{m}(\xi)$ by the given power $\left(1+e^{-i\xi}\right)^K$, the quotient is $\widehat{h}(\xi)$. The mask h may have the support $\mathcal{I}$.

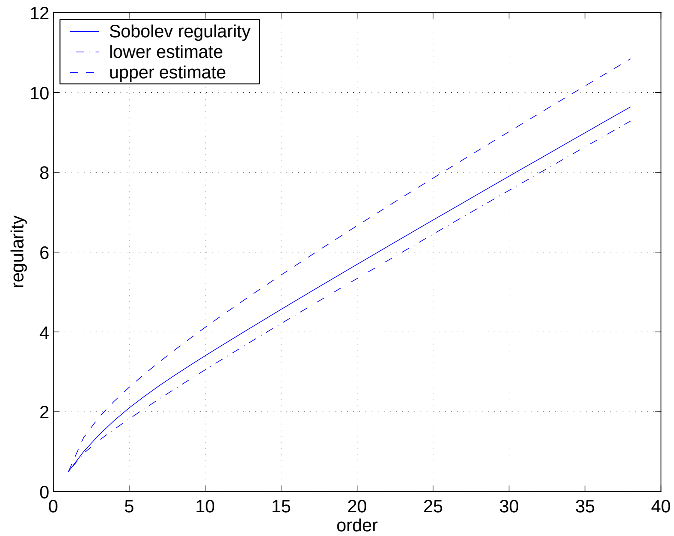

Fig. 1. SOBOLEV smoothness of DAUBECHIES wavelets (${}_N\phi$ as in [Dau92], ${}_1\phi$ is the HAAR generator) depending on the order of the wavelets.

3) Compute the sums $y_k = \sum_{j\in\mathbb{Z}} h_{k+3j}$.
4) Compute the square sum $p_2 = y_0^2 + y_1^2 + y_2^2$.
5) Compute $B_{h*h^*} = \sqrt{\frac{3}{2}\left(p_2 - \frac{1}{3}\right)^2 + \frac{1}{3}}$.
6) Eventually the SOBOLEV smoothness limit s_0 is bounded by

$$K - \log_4 2B_{h*h^*} \leq s_0$$

and further if one knows that the eigenvalues are all real then

$$s_0 \leq K - \log_4 2B_{h*h^*} + \frac{1}{2}\log_4\left(2 \cdot \#\mathcal{I} - 1\right).$$

Remark 4: Step 2 is numerical critical because the resulting filter has coefficients that vary heavily in magnitude, thus even simple criteria like the sum of the coefficients being 1 is infringed!

A. *Orthogonal* DAUBECHIES *wavelets*

For a given power of the factor $\left(1+e^{-i\xi}\right)$ in $\widehat{m}(\xi)$ (this is considered as the *order*) the DAUBECHIES wavelet filter is the shortest one that leads to an orthogonal wavelet basis. Actually there are several filters possible for one order but they all share the same filter $m*m^*$ and thus the same SOBOLEV smoothness. Figure 1 shows that the upper estimate of the smoothness is at most 1.5 too high and the lower estimate at most 0.5 too low.

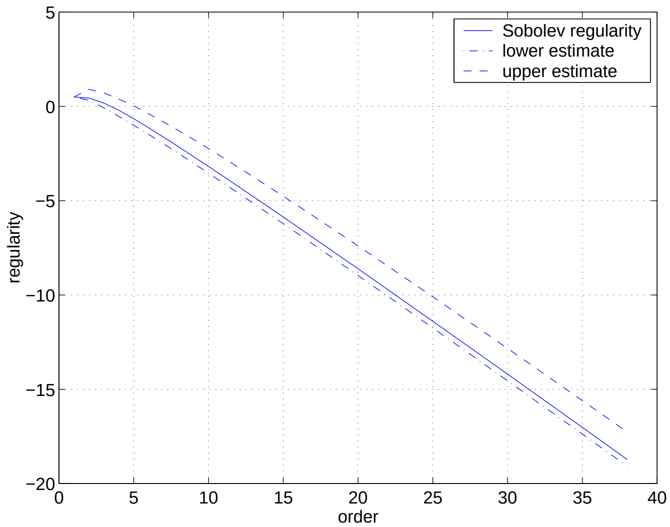

Fig. 2. SOBOLEV smoothness of the CDF primal generator ${}_{N,N}\phi$ depending on the order N.

B. Biorthogonal spline wavelets (CDF)

In contrast to orthogonal bases the CDF wavelet basis consists of two different generator functions, that are a primal and a dual generator. The dual generator ${}_{\widetilde{N}}\widetilde{\phi}$ is a $\widetilde{N}$th order B-spline, its Sobolev smoothness is $s_0 = \widetilde{N} - \frac{1}{2}$ and this is also the result of our estimate due to Theorem 3 since the filter consists only of a power of $\left(1 + e^{-i\xi}\right)$ and the eigenspectrum of the transition matrix of the remaining filter of length 1 will be estimated exactly.

That is why the more interesting function is the primal generator ${}_{\widetilde{N},N}\phi$ whose filter contains the Nth power of $\left(1 + e^{-i\xi}\right)$ and the remaining filter depends only on $\frac{N+\widetilde{N}}{2}$. The dependency on N is clear thus we content ourselves with the analysis of $\left({}_{N,N}\phi\right)_{N\in\mathbb{N}}$ which is a sequence of functions of decreasing smoothness as can be seen in Figure 2.

The maximum deviation from the lower bound is 0.4 and the deviation from the upper bound is at most 1.5.

APPENDIX

The following lemma gives a brief list of equivalences that are useful when dealing with operations on signals like convolution, upsampling, downsampling.

Lemma 10:

$$(h \uparrow k) \downarrow k = h \tag{3}$$
$$(h \uparrow k) \uparrow j = h \uparrow (k \cdot j)$$
$$(h \downarrow k) \downarrow j = h \downarrow (k \cdot j)$$
$$(g * h) \uparrow k = (g \uparrow k) * (h \uparrow k)$$
$$(g \uparrow k * h) \downarrow k = g * (h \downarrow k) \tag{4}$$

Remark 5: The identity (4) is an exception due to its asymmetry. The problem is that the distributivity with respect to down sampling, that is $(g * h) \downarrow k = (g \downarrow k) * (h \downarrow k)$, does not hold in general.

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